Highest weight representations of a Lie algebra of Block type[1]

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Abstract. For a field $\mathbb{F}$ of characteristic zero and an additive subgroup G of $\mathbb{F}$, a Lie algebra $\mathcal{B}(G)$ of Block type is defined with basis $\{L_{\alpha,i},\, c \,|\, \alpha\in G,\, -1\le i\in\mathbb{Z}\}$ and relations $[L_{\alpha,i},L_{\beta,j}]=((i+1)\beta-(j+1)\alpha)L_{\alpha+\beta,i+j}+\alpha\delta_{\alpha,-\beta}\delta_{i+j,-2}c$, $[c,L_{\alpha,i}]=0$. Given a total order $\succ$ on G compatible with its group structure, and any $\Lambda\in\mathcal{B}(G)_0^*$, a Verma $\mathcal{B}(G)$-module $M(\Lambda,\succ)$ is defined, and the irreducibility of $M(\Lambda,\succ)$ is completely determined. Furthermore, it is proved that an irreducible highest weight $\mathcal{B}(\mathbb{Z})$-module is quasifinite if and only if it is a proper quotient of a Verma module.

Key words: Verma modules, Lie algebras of Block type, irreducbility.

Mathematics Subject Classification (2000): 17B10, 17B65, 17B68.

§1. Introduction

Block [B] introduced a class of infinite dimensional simple Lie algebras over a field of characteristic zero. Generalizations of Block algebras (usually referred to as *Lie algebras of Block type*) have been studied by many authors (see, for example, [DZ, LT, S1, S2, X1, X2, WZ, ZM]). Partially because they are closely related to the Virasoro algebra (and some of them are sometimes called Virasoro-like algebras), these algebras have attracted some attention in the literature.

Let $\mathbb{F}$ be a field of characteristic 0 and G an additive subgroup of $\mathbb{F}$. The *Lie algebra* $\mathcal{B}(G)$ *of Block type* considered in this paper is the Lie algebra with basis $\{L_{\alpha,i}, c\,|\,\alpha\in G,\, i\in\mathbb{Z},\, i\ge -1\}$ and relations

$$[L_{\alpha,i},L_{\beta,j}]=((i+1)\beta-(j+1)\alpha)L_{\alpha+\beta,i+j}+\alpha\delta_{\alpha,-\beta}\delta_{i+j,-2}c,\quad [c,L_{\alpha,i}]=0. \tag{1.1}$$

Let

$$\mathcal{B}(G)_\alpha=\mathrm{span}\{L_{\alpha,i}|\ i\ge -1\}+\delta_{\alpha,0}\mathbb{F}c. \tag{1.2}$$

Then $\mathcal{B}(G)=\oplus_{\alpha\in G}\mathcal{B}(G)_\alpha$ is G-graded (but not finitely graded). Throughout this paper, we fix a total order "$\succ$" on G compatible with its group structure. Denote

$$G_+=\{x\in G\ \big|\ x\succ 0\},\quad G_-=\{x\in G\ \big|\ x\prec 0\}.$$

[1]Supported by a NSF grant 10471096 of China, "One Hundred Talents Program" from University of Science and Technology of China.

Then $G = G_+ \cup \{0\} \cup G_-$. Setting $\mathcal{B}(G)_\pm = \oplus_{\pm\alpha\succ 0}\mathcal{B}(G)_\alpha$, we have the triangular decomposition

$$\mathcal{B}(G) = \mathcal{B}(G)_- \oplus \mathcal{B}(G)_0 \oplus \mathcal{B}(G)_+.$$

Note that $\mathcal{B}(G)_0 = \mathrm{span}\{L_{0,i} \mid i \geq -1\} \oplus \mathbb{F}c$ is a commutative subalgebra of $\mathcal{B}(G)$ (but it is not a Cartan subalgebra).

A $\mathcal{B}(G)$-module V is *quasifinite* if V is finitely G-graded, namely,

$$V = \oplus_{\alpha\in G} V_\alpha, \quad \mathcal{B}(G)_\alpha V_\beta \subset V_{\alpha+\beta}, \quad \mathrm{dim} V_\alpha < \infty \quad \text{for} \quad \alpha, \beta \in G.$$

Quasifinite modules are closed studied by some authors, e.g., [KL, KR, S1, S2]. In [S1], it is proved that a quasifinite irreducible $\mathcal{B}(\mathbb{Z})$-module is a highest or lowest weight module and the quasifinite irreducible highest weight modules are classified. The main result of this paper is the following.

Theorem 1.1 (1) *An irreducible highest weight $\mathcal{B}(\mathbb{Z})$-module is quasifinite if and only if it is a proper quotient of a Verma module.*

(2) *Let $\Lambda \in \mathcal{B}(G)_0^*$. With respect to a dense order "$\succ$" of G (cf. (2.3)), the Verma $\mathcal{B}(G)$-module $M(\Lambda, \succ)$ is irreducible if and only if $\Lambda \neq 0$. Moreover, in case $\Lambda = 0$, if we set*

$$M'(0, \succ) = \sum_{k>0,\, \alpha_1,\cdots,\alpha_k\in G_+} \mathbb{F} L_{-\alpha_1,i_1} \cdots L_{\alpha_k,i_k} v_0,$$

then $M'(0, \succ)$ is an irreducible submodule of $M(0, \succ)$ if and only if for all $x, y \in G_+$, there exists a positive integer n such that $nx \succ y$.

(3) *With respect to a discrete order "$\succ$" (cf. (2.4)), the Verma $\mathcal{B}(G)$-module $M(\Lambda, \succ)$ is irreducible if and only if $M_a(\Lambda, \succ)$ is an irreducible $\mathcal{B}(a\mathbb{Z})$-module.*

§2. Verma modules over $\mathcal{B}(G)$

Let $U = U(\mathcal{B}(G))$ be the universal enveloping algebra of $\mathcal{B}(G)$. For any $\Lambda \in \mathcal{B}(G)_0^*$ (the dual space of $\mathcal{B}(G)_0$), let $I(\Lambda, \succ)$ be the left ideal of U generated by the elements

$$\{L_{\alpha,i} \mid \alpha \succ 0, i \geq -1\} \cup \{h - \Lambda(h) \cdot 1 \mid h \in \mathcal{B}(G)_0\}.$$

Then the *Verma $\mathcal{B}(G)$-module* with respect to the order "$\succ$" is defined as

$$M(\Lambda, \succ) = U/I(\Lambda, \succ).$$

By the PBW theorem, it has a basis consisting of all vectors of the form

$$L_{-\alpha_1,i_1} L_{-\alpha_2,i_2} \cdots L_{-\alpha_k,i_k} v_\Lambda,$$

where v_Λ is the coset of 1 in $M(\Lambda, \succ)$, and

$$-1 \leq i_j \in \mathbb{Z}, \quad 0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_k, \quad \text{and} \;\; i_s \leq i_{s+1} \;\; \text{if} \;\; \alpha_s = \alpha_{s+1}.$$

Note that $M(\Lambda, \succ)$ is a highest weight $\mathcal{B}(G)$-module in the sense that $M(\Lambda, \succ) = \oplus_{\alpha \preccurlyeq 0} M_\alpha$, where $M_0 = \mathbb{F} v_\Lambda$, and M_α for $\alpha \prec 0$ is spanned by

$$L_{-\alpha_1, i_1} L_{-\alpha_2, i_2} \cdots L_{-\alpha_k, i_k} v_\Lambda, \tag{2.1}$$

with $i_j \geq -1$, $0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_k$, and $\alpha_1 + \cdots + \alpha_k = -\alpha$. Thus $M(\Lambda, \succ)$ is a G-graded $\mathcal{B}(G)$-module with $\dim M_{-\alpha} = \infty$ for any $\alpha \in G_+$.

We call a nonzero vector $u \in M_\alpha$ a *weight vector with weight* α. For any $a \in G$, denote

$$\mathcal{B}(a\mathbb{Z}) = \text{span}\{L_{na,k} \mid a \in \mathbb{Z}, k \geq -1\},$$

a subalgebra of $\mathcal{B}(G)$ isomorphic to $\mathcal{B}(\mathbb{Z})$. We also denote

$$M_a(\Lambda, \succ) \quad = \quad \mathcal{B}(a\mathbb{Z})\text{-submodule of } M(\Lambda, \succ) \text{ generated by } v_\Lambda.$$

Denote

$$B(\alpha) = \{\beta \in G \mid 0 \prec \beta \prec \alpha\} \quad \text{for} \;\; \alpha \in G_+. \tag{2.2}$$

The order " $\succ$" is called *dense* if

$$\#B(\alpha) = \infty \quad \text{for all} \;\; \alpha \in G_+, \tag{2.3}$$

it is *discrete* if

$$B(a) = \emptyset \quad \text{for some} \;\; a \in G_+. \tag{2.4}$$

Proof of Theorem 1.1(2) and (3). (2) Suppose the order " $\succ$" is dense. For each $m \in \mathbb{N} = \{1, 2, ...\}$, set

$$V_m = \sum_{\substack{0 \leq s \leq m, \; i_1, \cdots i_s \geq -1 \\ 0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_s}} \mathbb{F} L_{-\alpha_1, i_1} \cdots L_{-\alpha_s, i_s} v_\Lambda, \tag{2.5}$$

where $i_s \leq i_{s+1}$ if $\alpha_s = \alpha_{s+1}$. It is clear that $L_{\alpha,k} V_m \subseteq V_m$ for any $\alpha \in G_+$, $k \geq -1$.

Let $u_0 \neq 0$ be any given weight vector in $M(\Lambda, \succ)$. We want to prove that $v_\Lambda \in U(\mathcal{B}(G)) u_0$ if $\Lambda \neq 0$. We divide the proof into four steps:

Step 1. We claim that there exists some weight vector $u \in U(\mathcal{B}(G)) u_0$ such that, for some $r \in \mathbb{N}$,

$$u \equiv \sum_{k_1, \cdots, k_r \in \mathbb{N}} a_{\underline{k}} L_{-\varepsilon_r, k_r} \cdots L_{-\varepsilon_1, k_1} v_\Lambda \;(\text{mod } V_{r-1}) \quad \text{for some} \;\; a_{\underline{k}} \in \mathbb{F},$$

where $0 \prec \varepsilon_r \prec \cdots \prec \varepsilon_1$, and $0 \neq a_{\underline{k}} \in \mathbb{F}$ for some $\underline{k} = (k_r, \cdots, k_1)$.

It is clear that $u_0 \in V_r \setminus V_{r-1}$ for some $r \in \mathbb{N}$. If $r \leq 1$, our claim clearly holds. So we assume that $r > 1$. Hence we can write

$$u_0 \equiv \sum_{0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_r} a_{\underline{\alpha},\underline{i}} L_{-\alpha_1,i_1} \cdots L_{-\alpha_r,i_r} v_\Lambda \ (\mathrm{mod}\ V_{r-1}) \quad \text{for some} \quad a_{\underline{\alpha},\underline{i}} \in \mathbb{F},$$

where $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$, $\underline{i} = (i_1, ..., i_r)$, and we denote

$$(\underline{\alpha}, \underline{i}) = (\alpha_1, ..., \alpha_r, i_1, ..., i_r).$$

Let $I = \{(\underline{\alpha}, \underline{i}) \,|\, a_{\underline{\alpha},\underline{i}} \neq 0\}$. By assumption, $I \neq \emptyset$. For any $\underline{\alpha}$ and $\underline{\alpha}' = (\alpha'_1, \cdots, \alpha'_r)$, we define

$$\underline{\alpha} \succ \underline{\alpha}' \quad \Longleftrightarrow \quad \exists\, s \in \{1, ..., r\} \ \text{ such that } \ \alpha_s \succ \alpha'_s, \ \text{ and } \ \alpha_t = \alpha'_t \ \text{ for } \ t > s. \tag{2.6}$$

Similarly, for any $\underline{i}$ and $\underline{i}' = (i'_1, \cdots, i'_r)$, we define

$$\underline{i} > \underline{i}' \quad \Longleftrightarrow \quad \exists\, s \in \{1, ..., r\} \ \text{ such that } \ i_s > i_s, \ \text{ and } \ i_t = i'_t \ \text{ for } \ t > s. \tag{2.7}$$

For any $(\underline{\alpha}, \underline{i})$, $(\underline{\alpha}', \underline{i}') \in I$, we define

$$(\underline{\alpha}, \underline{i}) \succ (\underline{\alpha}', \underline{i}') \quad \Longleftrightarrow \quad \underline{\alpha} \succ \underline{\alpha}', \quad \text{or} \quad \underline{\alpha} = \underline{\alpha}', \ \underline{i} > \underline{i}'. \tag{2.8}$$

Let

$$(\underline{\beta}, \underline{j}) = (\beta_1, \cdots, \beta_r, j_1, \cdots, j_r), \quad 0 \prec \beta_1 \preccurlyeq \cdots \preccurlyeq \beta_r,$$

be the unique maximal element in I. Then we can write β as

$$\beta = (\beta_1, ..., \beta_s, \beta_r, ..., \beta_r) \quad \text{for some} \quad s \in \{1, ..., r\}.$$

By the assumption that "$\succ$" is a dense order, we can always find some $\varepsilon_1 \in G_+$ such that

$$\varepsilon_1 \prec \beta_1 \ \text{ and } \ \{x \in G_+ \,|\, \beta_r - \varepsilon_1 \prec x \prec \beta_r\} \cap \{\alpha_r, \alpha_{r-1} \,|\, (\underline{\alpha}, \underline{i}) \in I \text{ for some } \underline{i}\} = \emptyset.$$

Using relations (1.1), and noting that $\beta_r - \varepsilon_1 - \alpha_k \succ 0$ if $\alpha_k \neq \beta_r, k \in \{1, \cdots, r\}$, by choosing $k_1 \gg 0$ with $k_1 \in \mathbb{N}$, we see that

$$u_1 := L_{\beta_r - \varepsilon_1, k_1} u_0 \equiv \sum_{0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_{r-1},\, k'_1 \in \mathbb{N}} a^{(1)}_{\underline{\alpha},\underline{j}} L_{-\varepsilon_1, k'_1} L_{-\alpha_1, i_1} \cdots L_{-\alpha_{r-1}, i_{r-1}} v_\Lambda \ (\mathrm{mod}\ V_{r-1}),$$

for some $a^{(1)}_{\underline{\alpha},\underline{j}} \in \mathbb{F}$. Set

$$I^{(1)} = \big\{ \big(\varepsilon_1, \alpha_1, \cdots, \alpha_{r-1}, k'_1, i_1, \cdots, i_{r-1}\big) \ \big| \ a^{(1)}_{\underline{\alpha},\underline{i}} \neq 0 \big\}.$$

The coefficient corresponding to

$$(\underline{\beta}^{(1)}, \underline{j}^{(1)}) = (\varepsilon_1, \beta_1, \cdots \beta_{r-1}, k_1 + j_{s+1}, j_1, \cdots, j_{r-1})$$

$((\underline{\beta}^{(1)}, \underline{j}^{(1)})$ maybe not the maximal element in $I^{(1)})$ is

$$-m((k_1+1)\beta_r + (j_{s+1}+1)(\beta_r - \varepsilon_1))a_{\underline{\beta},\underline{j}} \neq 0 \quad \text{for some} \;\; m \in \mathbb{N}.$$

Thus $I^{(1)} \neq \emptyset$.

Now for $p = 2, \cdots, r$, we define recursively and easily prove by induction that

(i) Let $\varepsilon_p \in G_+$ such that $\varepsilon_p \prec \varepsilon_{p-1}$ and

$$\big\{x \in G \,|\, \beta_{r-p+1} - \varepsilon_p \prec x \prec \beta_{r-p+1}\big\} \cap \big\{\alpha_{r-p+1}, \alpha_{r-p} \,|\, (\underline{\alpha}, \underline{j}) \in I^{(p-1)}\big\} = \emptyset.$$

(ii) Choose $k_p \gg 0$ and let $u_p = L_{\beta_{r-p+1}-\varepsilon_p, k_p} u_{p-1}$. Then, for some $a^{(p)}_{\underline{\alpha},\underline{j}} \in \mathbb{F}$,

$$u_p \equiv \sum_{0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_{r-p}} a^{(p)}_{\underline{\alpha},\underline{j}} L_{-\varepsilon_p, k'_p} \cdots L_{-\varepsilon_1, k'_1} L_{-\alpha_1, i_1} \cdots L_{-\alpha_{r-p}, i_{r-p}} v_\Lambda \;(\mathrm{mod}\; V_{r-1}).$$

(iii) Let

$$I^{(p)} = \big\{(\varepsilon_p, \cdots, \varepsilon_1, \alpha_1, \cdots, \alpha_{r-p}, k'_p, \cdots, k'_1, i_1, \cdots, j_{r-p}) \,|\, a^{(p)}_{\underline{\alpha},\underline{j}} \neq 0\big\}.$$

Then $I^{(p)} \neq \emptyset$.

Now our claim follows immediately by letting $p = r$.

Step 2. We claim that there exists some weight vector $u \in U(\mathcal{B}(G))u_0$ such that, for some $r \in \mathbb{N}$,

$$u \equiv L_{-\varepsilon_r, k_r} \cdots L_{-\varepsilon_1, k_1} v_\Lambda \;(\mathrm{mod}\; V_{r-1}),$$

where $k_j \geq -1$, and $0 \prec \varepsilon_r \prec \cdots \prec \varepsilon_1$.

By Step 1, there exists some weight vector $u \in U(\mathcal{B}(G))u_0$ such that, for some $r \in \mathbb{N}$,

$$u \equiv \sum_{k_1, \cdots, k_r \in \mathbb{N}} a_{\underline{k}} L_{-\varepsilon_r, k_r} \cdots L_{-\varepsilon_1, k_1} v_\Lambda \;(\mathrm{mod}\; V_{r-1}) \quad \text{for some} \;\; a_{\underline{k}} \in \mathbb{F},$$

where $0 \prec \varepsilon_r \prec \cdots \prec \varepsilon_1$ and

$$K := \{\underline{k} = (k_r, \cdots, k_1) \,|\, a_{\underline{k}} \neq 0\} \neq \emptyset.$$

Let

$$\underline{j} = (-1, \cdots, -1, j_s, \cdots, j_1) \quad \text{with} \;\; j_s \neq -1,$$

be the unique maximal element in K (recall (2.7)). Assume that K is not a singleton. Then $\underline{j} \neq (-1, \cdots, -1)$. Set

$$\delta = \min\{\{\varepsilon_i, \varepsilon_j - \varepsilon_i \,|\, 1 \leq j < i \leq r\} \cap G_+\}.$$

Let $\varepsilon' \in G_+$ such that $\varepsilon' \prec \delta$. Then

$$\begin{aligned} L_{\varepsilon',-1} \cdot u \quad &\equiv \quad \sum_{k_r, \cdots, k_1 \in \mathbb{N}} \sum_{j=0}^{r} -(k_j+1)\varepsilon' a_{\underline{k}} L_{-\varepsilon_r,k_r} \cdots L_{-\varepsilon_{j+1},k_{j+1}} \\ &\qquad \times L_{\varepsilon'-\varepsilon_j,k_j-1} L_{-\varepsilon_{j-1},k_{j-1}} \cdots L_{-\varepsilon_1,k_1} v_\Lambda \pmod{V_{r-1}}. \end{aligned}$$

The term

$$L_{-\varepsilon_r,-1} \cdots L_{-\varepsilon_{s+1},-1} L_{\varepsilon'-\varepsilon_s,j_s-1} L_{-\varepsilon_{s-1},j_{s-1}} \cdots L_{-\varepsilon_1,j_1} v_\Lambda$$

appears in $L_{\varepsilon,-1} \cdot u$, since the corresponding coefficient is $-(j_s+1)a_{\underline{j}} \neq 0$. Using the same arguments as above and the induction on $\max\{k_r + \cdots + k_1 \,|\, a_{\underline{k}} \neq 0\}$, we see that there exists some weight vector $u \in U(\mathcal{B}(G))u_0$ such that

$$u \equiv \sum_{0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_r} a_{\underline{\alpha}} L_{-\alpha_1,-1} \cdots L_{-\alpha_r,-1} v_\Lambda \pmod{V_{r-1}} \quad \text{for some} \quad a_{\underline{\alpha}} \in \mathbb{F}.$$

Using the same arguments as in Step 1, we can prove the claim.

Step 3. We claim that there exists some $\varepsilon \in G_+$ such that $L_{-\varepsilon,k} v_\Lambda \in U(\mathcal{B}(G))u_0$ for $k \geq -1$.

By Step 2, there is a weight vector $u \in U(\mathcal{B}(G))u_0$ such that

$$u = L_{-\varepsilon_r,k_r} \cdots L_{-\varepsilon_1,k_1} v_\Lambda + \sum_{0 \leq l < r,\, 0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_l} b_{\underline{\alpha},\underline{i}} L_{-\alpha_1,i_1} \cdots L_{-\alpha_l,i_l} v_\Lambda,$$

for some $b_{\underline{\alpha},\underline{i}} \in \mathbb{F}$, where $0 \prec \varepsilon_r \prec \cdots \prec \varepsilon_1$. Assume that $u \notin \mathbb{F} v_\Lambda$.

Set

$$I_0 = \{\underline{\alpha} = (\alpha_1, \cdots, \alpha_l) \,|\, b_{\underline{\alpha},\underline{i}} \neq 0 \ \text{ for some } \ \underline{i}\}, \quad \underline{j}^{(0)} = \min\{\varepsilon_r, \alpha_1 \,|\, \underline{\alpha} \in I_0\}.$$

Let $\varepsilon \in G_+$ such that $\varepsilon \prec \underline{j}^{(0)}$. Assume that $u \in M_\lambda$ (cf. (2.1)). By relations (1.1), we have

$$\begin{aligned} L_{-\lambda-\varepsilon,j} u \quad &= \quad f(-\lambda-\varepsilon) L_{-\varepsilon,j+k_r+\cdots+k_1} v_\Lambda + \sum_{1 \leq l < r,\, 0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_l} b_{\underline{\alpha},\underline{i}} g_{\underline{\alpha},\underline{i}}(-\lambda-\varepsilon) L_{-\varepsilon,j+i_1+\cdots+i_l} v_\Lambda \\ &= \quad \Big\{ f(-\lambda-\varepsilon) + \sum_{\substack{1 \leq l < r,\, 0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_l \\ i_1+\cdots+i_l = k_1+\cdots+k_r}} b_{\underline{\alpha},\underline{i}} g_{\underline{\alpha},\underline{i}}(-\lambda-\varepsilon) \Big\} L_{-\varepsilon,j+k_r+\cdots+k_1} v_\Lambda \\ &\qquad + \sum_{\substack{1 \leq l < r,\, 0 \prec \alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_l \\ i_1+\cdots+i_l \neq k_1+\cdots+k_r}} b_{\underline{\alpha},\underline{i}} g_{\underline{\alpha},\underline{i}}(-\lambda-\varepsilon) L_{-\varepsilon,j+i_1+\cdots+i_l} v_\Lambda \quad \in \quad U(\mathcal{B}(G))u_0, \end{aligned}$$

where in general $f(x)$ and $g_{\underline{a},\underline{i}}(x)$ are determinants:

$$f(x) = \begin{vmatrix} j+1 & k_r+1 \\ x & -\varepsilon_r \end{vmatrix} \begin{vmatrix} j+k_r+1 & k_{r-1}+1 \\ x-\varepsilon_r & -\varepsilon_{r-1} \end{vmatrix} \cdots \begin{vmatrix} j+k_r+\cdots+k_2+1 & k_1+1 \\ x-\varepsilon_r-\cdots-\varepsilon_2 & -\varepsilon_1 \end{vmatrix},$$

$$g_{\underline{a},\underline{i}}(x) = \begin{vmatrix} j+1 & i_1+1 \\ x & -\alpha_1 \end{vmatrix} \begin{vmatrix} j+i_1+1 & i_2+1 \\ x-\alpha_1 & -\alpha_2 \end{vmatrix} \cdots \begin{vmatrix} j+i_1+\cdots+i_{k-1}+1 & i_k+1 \\ x-\alpha_1-\cdots-\alpha_{k-1} & -\alpha_k \end{vmatrix},$$

Since $\deg f(x) = r > \deg g_{\underline{\alpha},\underline{i}}(x)$ for all $\underline{\alpha} \in I_0$, we can find $\varepsilon \in G_+$ with $\varepsilon \prec \underline{j}^{(0)}$ such that

$$f(-\lambda-\varepsilon) + \sum_{1\le l<r,\, 0\prec\alpha_1\preccurlyeq\cdots\preccurlyeq\alpha_l} b_{\underline{\alpha},\underline{i}}\, g_{\underline{\alpha},\underline{i}}(\lambda-\varepsilon) \neq 0.$$

So we obtain some vector

$$u = (a_1 L_{-\varepsilon,i_1} + \cdots + a_n L_{-\varepsilon,i_n})v_\Lambda \in U(\mathcal{B}(G))u_0,$$

for some $0 \neq a_1, \cdots, a_n \in \mathbb{F}$. Choosing $\varepsilon' \in G_+$ with $\varepsilon' \prec \varepsilon$, using

$$L_{\varepsilon-\varepsilon',-1}u = -(\varepsilon-\varepsilon')(a_1(i_1+1)L_{-\varepsilon',i_1-1} + \cdots + a_n(i_n+1)L_{-\varepsilon',i_n-1})v_\Lambda \in U(\mathcal{B}(G))u_0,$$

and induction on $\max\{i_1, \cdots, i_n\}$, one can deduce that there exists some $\varepsilon' \prec \varepsilon$ such that $L_{-\varepsilon',-1}v_\Lambda \in U(\mathcal{B}(G))u_0$. Let $\varepsilon'' \in G_+$ such that $\varepsilon'' \prec \varepsilon'$. Then

$$L_{-\varepsilon'',k-1}v_\Lambda = -((k+1)\varepsilon')^{-1} L_{\varepsilon'-\varepsilon'',k} L_{-\varepsilon',-1}v_\Lambda \in U(\mathcal{B}(G))u_0 \quad \text{for all} \quad k \ge 0.$$

This proves our claim.

Step 4. We claim that if there exists some $\varepsilon \in G_+$ such that $L_{-\varepsilon,k}v_\Lambda \in U(\mathcal{B}(G))u_0$ for all $k \ge -1$, then $L_{-x,k}v_\Lambda \in U(\mathcal{B}(G))u_0$ for all $k \ge -1$ and all $x \in B'(\varepsilon)$, where $B'(\varepsilon)$ is defined by

$$B'(\varepsilon) = \mathrm{span}_{\mathbb{Z}_+}\{y \in G_+ \mid y \preccurlyeq \varepsilon\}.$$

Let $\varepsilon' \in G_+$ such that $\varepsilon' \prec \varepsilon$. Then

$$L_{-\varepsilon',k-1}v_\Lambda = -((k+1)(\varepsilon-\varepsilon'))^{-1} L_{\varepsilon-\varepsilon',-1} L_{-\varepsilon,k}v_\Lambda \in U(\mathcal{B}(G))u_0.$$

Since

$$(k+1)\varepsilon' L_{-(\varepsilon+\varepsilon'),k-1}v_\Lambda = L_{-\varepsilon',-1}L_{-\varepsilon,k}v_\Lambda - L_{-\varepsilon,k}L_{-\varepsilon',-1}v_\Lambda \in U(\mathcal{B}(G))u_0,$$

it follows that $L_{-(\varepsilon'+\varepsilon),k}v_\Lambda \in U(\mathcal{B}(G))u_0$ for all $k \ge -1$. Similarly, we deduce that

$$L_{-x,k}v_\Lambda \in U(\mathcal{B}(G))u_0 \quad \text{for all} \quad k \ge -1 \quad \text{and all} \quad x \in \mathbb{Z}_+\varepsilon + \mathbb{Z}_+\varepsilon'.$$

Our claim follows.

By Step 3, we have $L_{-\varepsilon',-1}v_\Lambda \in U(\mathcal{B}(G))u_0$ for some $\varepsilon' \in G_+$. From

$$\begin{aligned} L_{\varepsilon',-1}L_{-\varepsilon',-1}v_\Lambda &= \varepsilon' c \cdot v_\Lambda = \Lambda(c)v_\Lambda, \\ L_{\varepsilon',k}L_{-\varepsilon',-1}v_\Lambda &= -(k+1)\varepsilon' L_{0,k-1}v_\Lambda = -(k+1)\varepsilon'\Lambda(L_{0,k-1})v_\Lambda \text{ for } k \geq 0, \end{aligned}$$

it is easy to see that $v_\Lambda \in U(\mathcal{B}(G))u_0$ if $\Lambda \neq 0$, hence in this case, $M(\Lambda, \succ)$ is irreducible.

On the other hand, if $\Lambda = 0$, then it is clear that

$$M'(0,0) = \sum_{k>0,\, \alpha_1,\cdots,\alpha_k \in G_+} \mathbb{F}L_{-a_1,i_1}\cdots L_{-\alpha_k,i_k}v_0$$

is a proper $U(\mathcal{B}(G))$-submodule. Assume that for all $x, y \in G_+$ there exists a positive integer n such that $nx \succcurlyeq y$. By Steps 1–4, there exists $\varepsilon' \in G_+$ such that $L_{-n\varepsilon',-1}v_0 \in M'(0,0)$ for all $n \in \mathbb{N}$. Thus for any $z \in G_+$, using $y = n\varepsilon' \succcurlyeq z$ for some $n \in \mathbb{N}$, we have

$$L_{-z,k-1}v_0 = -((k+1)y)^{-1}L_{y-z,k}L_{-y,-1}v_0 \in M'(0,0) \quad \text{for all} \ \ k \in \mathbb{Z}_+.$$

We see that $M'(0,0)$ is in fact an irreducible $\mathcal{B}(G)$-module.

If there exists $x, y \in G_+$ such that $\mathbb{N}x \prec y$, then $B(x) \prec y$ (cf. (2.2)). It is easy to verify that

$$W' = U(\mathcal{B}(G))\{L_{-z,k} \mid z \in B(x), k \geq -1\}v_\Lambda,$$

is a proper submodule of $M'(0,0)$ since $L_{-y}v_\Lambda \notin W'$.

(2) Suppose the order "$\succ$" is discrete (recall (2.4)). Note that $a\mathbb{Z} \subseteq G$. For any $x \in G$, we write $x \succ a\mathbb{Z}$ if $x \succ na$ for any $n \in \mathbb{Z}$. Let

$$H_+ = \big\{x \in G \bigm| x \succ a\mathbb{Z}\big\}, \quad H_- = -H_+.$$

Denote by $\mathcal{B}(H_+)$ the subalgebra of $\mathcal{B}(G)$ generated by $\{L_{\alpha,k} \,|\, \alpha \in H_+, k \geq -1\}$. It is not difficult to see that $G = a\mathbb{Z} \cup H_+ \cup H_-$. Obviously, $\mathcal{B}(H_+)M_a(\Lambda, \succ) = 0$. Since

$$M(\Lambda, \succ) \cong U(\mathcal{B}(G)) \otimes_{U(\mathcal{B}(a\mathbb{Z})+\mathcal{B}(H_+))} M_a(\Lambda, \succ),$$

it follows that the irreducibility of $\mathcal{B}(G)$-module $M(\Lambda, \succ)$ imply the irreducibility of $\mathcal{B}(a\mathbb{Z})$-module $M_a(\Lambda, \succ)$.

Conversely, suppose $M_a(\Lambda, \succ)$ is an irreducible $\mathcal{B}(a\mathbb{Z})$-module. Let $u_0 \notin \mathbb{F}v_\Lambda$ be any given weight vector in $M(\Lambda, \succ)$. Then $u_0 \in V_r \setminus V_{r-1}$ for some $r \in \mathbb{N}$. We want to prove that

$U(\mathcal{B}(G))u_0 \cap M_a(\Lambda, \succ) \neq \{0\}$, from which the irreducibility of $M(\Lambda, \succ)$ as a $\mathcal{B}(G)$-module follows immediately.

Write

$$u_0 \equiv \sum_{\substack{\alpha'_1,\cdots,\alpha'_s \in H_+,\, \alpha_1,\cdots,\alpha_{r-s} \in a\mathbb{Z}_+, \\ \alpha'_1 \succcurlyeq \cdots \succcurlyeq \alpha'_s,\, \alpha_1 \succcurlyeq \cdots \succcurlyeq \alpha_{r-s}}} a_{\bar{\alpha},\bar{j}} L_{-\alpha'_1,j'_1} \cdots L_{-\alpha'_s,j'_s} L_{-\alpha_1,j_1} \cdots L_{-\alpha_{r-s},j_{r-s}} v_\Lambda \ (\mathrm{mod}\ V_{r-1}),$$

for some $a_{\bar{\alpha},\bar{j}} \in \mathbb{F}$, where $j_t \geq j_{t+1}$ if $\alpha_t = \alpha_{t+1}$, and $j'_t \geq j'_{t+1}$ if $\alpha'_t = \alpha'_{t+1}$, and

$$(\bar{\alpha}, \bar{j}) = (\alpha'_1, \cdots, \alpha'_s, \alpha_1, \cdots, \alpha_{r-s}, j'_1, \cdots, j'_s, j_1, \cdots, j_{r-s}).$$

Let

$$\bar{I} = \big\{(\alpha'_1, \cdots, \alpha'_s, \alpha_1, \cdots, \alpha_{r-s}, j'_1, \cdots, j'_s, j_1, \cdots, j_{r-s}) \,|\, a_{\bar{\alpha},\bar{j}} \neq 0\big\}.$$

By assumption, $\bar{I} \neq \emptyset$.

For any

$$\begin{aligned} (\bar{\alpha}, \bar{j}) &= (\alpha'_1, \cdots, \alpha'_s, \alpha_1, \cdots, \alpha_{r-s}, j'_1, \cdots, j'_s, j_1, \cdots, j_{r-s}) \in \bar{I}, \\ (\bar{\gamma}, \bar{l}) &= (\gamma'_1, \cdots, \gamma'_t, \gamma_1, \cdots, \gamma_{r-t}, l'_1, \cdots, l'_t, l_1, \cdots, l_{r-t}) \in \bar{I}, \end{aligned}$$

we define $(\bar{\alpha}, \bar{j}) \succ' (\bar{\gamma}, \bar{l})$ if and only if (cf. (2.6)–(2.8))

$$(\alpha_{r-s}, \cdots, \alpha_1, \alpha'_s, \cdots, \alpha'_1, j_{r-s}, \cdots, j_1, j'_s, \cdots, j'_1) \succ (\gamma_{r-t}, \cdots, \gamma_1, \gamma'_t, \cdots, \gamma'_1, l_{r-t}, \cdots, l_1, l'_t, \cdots, l'_1).$$

Let

$$(\bar{\beta}, \bar{i}) = (\beta'_1, \cdots, \beta'_1, \beta'_{t+1}, \cdots, \beta'_m, \beta_1, \cdots, \beta_{r-m}, i'_1, \cdots, i'_t, i'_{t+1}, \cdots, i'_m, i_1, \cdots, i_{r-m})$$

with $\beta'_1 \neq \beta'_{t+1}$, be the unique maximal element in $\bar{I}$ with respect to $\succ'$. Note that $\beta'_1 - \alpha'_k - a \succcurlyeq 0$ if $\beta'_1 \neq \alpha'_k$. Then for $k_1 \gg 0$, we have

$$\begin{aligned} u(1) := L_{\beta'_1 - a, k_1} u_0 \equiv \sum_{\substack{\alpha'_1,\cdots,\alpha'_{s-1} \in H_+,\, \alpha_1,\cdots,\alpha_{r-s} \in a\mathbb{Z}_+, \\ \alpha'_1 \succcurlyeq \cdots \succcurlyeq \alpha'_{s-1},\, \alpha_1 \succcurlyeq \cdots \succcurlyeq \alpha_{r-s},\, k'_1 \in \mathbb{N}}} & a^{(1)}_{\bar{\alpha},\bar{j}} L_{-\alpha'_1,j'_1} \cdots L_{-\alpha'_{s-1},j'_{s-1}} \\ & \times L_{-a,k'_1} L_{-\alpha_1,j_1} \cdots L_{-\alpha_{r-s},j_{r-s}} v_\Lambda \quad (\mathrm{mod}\, V_{r-1}), \end{aligned}$$

for some $a^{(1)}_{\bar{\alpha},\bar{j}} \in \mathbb{F}$. Set

$$\bar{I}^{(1)} = \{(\alpha'_1, \cdots, \alpha'_{s-1}, a, \alpha_1, \cdots, \alpha_{r-s}, j'_1, \cdots, j'_{s-1}, k'_1, j_1, \cdots, j_{r-s}) \,|\, a^{(1)}_{\bar{\alpha},\bar{j}} \neq 0\}.$$

$$(\bar{\beta}, \bar{i})^{(1)} = (\beta'_1, \cdots, \beta'_1, \beta'_{t+1}, \cdots, \beta'_m, a, \beta_1, \cdots, \beta_{r-m}, i'_1, \cdots, i'_{t-1}, i'_{t+1}, \cdots, i'_m, k_1 + i'_t, i_1, \cdots, i_{r-m}).$$

The term

$$L_{-\beta'_1,i'_1}\cdots L_{-\beta'_1,i'_{t-1}}L_{-\beta'_{t+1},i'_{t+1}}\cdots L_{-\beta'_m,i'_m}L_{a,k_1+i'_t}L_{\beta_1,i_1}\cdots L_{-\beta_{r-m},i_{r-m}}v_\Lambda$$

appears in $u(1)$ since the corresponding coefficient is

$$-m((k_1+1)\beta'_1+(i'_s+1)(\beta'_1-a))a_{\bar{\beta},\bar{i}}\neq 0 \quad \text{for some} \ \ m\in\mathbb{N}.$$

Thus $\bar{I}^{(1)}\neq\emptyset$. Now for $l=2,\cdots,r$, we define recursively and prove by induction that

(i) Choose $k_l\gg 0$ and let $u(l)=L_{\beta'_{m-l+1}-a,k_l}u(l-1)$. Then

$$\begin{aligned} u(s) \ \ &\equiv \sum_{\substack{k'_1,\cdot,k'_l\in\mathbb{N},\,\alpha'_1,\cdots,\alpha'_{s-l}\in H_+,\\ \alpha_1,\cdots,\alpha_{r-s}\in a\mathbb{Z}_+}} a^{(l)}_{\bar{\alpha},\bar{j}}L_{-\alpha'_1,j'_1}\cdots L_{-\alpha'_{s-l},j'_{s-l}} \\ &\qquad \times\, L_{-a,k'_l}\cdots L_{-a,k'_1}L_{-\alpha_1,j_1}\cdots L_{-\alpha_{r-s},j_{r-s}}v_\Lambda \ (\mathrm{mod}\ V_{r-1}), \end{aligned}$$

for some $a^{(l)}_{\bar{\alpha},\bar{j}}\in\mathbb{F}$, where $\alpha'_1\succcurlyeq\cdots\succcurlyeq\alpha'_{s-l}$, $\alpha_1\succcurlyeq\cdots\succcurlyeq\alpha_{r-s}$.

(ii) Let

$$\bar{I}^{(l)}=\{(\alpha'_1,\cdots,\alpha'_{s-l},a,\cdots,a,\alpha_1,\cdots,\alpha_{r-s},j'_1,\cdots,j'_{s-l},k'_l,\cdots,k'_1,j_1,\cdots,j_{r-s})\,|\,a^{(s)}_{\bar{\alpha},\bar{j}}\neq 0\}.$$

Then $\bar{I}^{(l)}\neq\emptyset$.

Now letting $l=m$ and noting that $u(m)$ is a weight vector, we obtain that $0\neq u(m)\in U(\mathcal{B}(G))u_0\cap M_a(\Lambda,\succ)$ as required. $\square$

§3. Verma modules over $\mathcal{B}(\mathbb{Z})$

Following [S1], we realize the Lie algebra $\mathcal{B}(\mathbb{Z})$ in the space $\mathbb{F}[x,x^{-1},t]\oplus\mathbb{F}c$ with the bracket

$$[x^\alpha f(t),x^\beta g(t)]=x^{\alpha+\beta}(\beta f'(t)g(t)-\alpha f(t)g'(t))+\alpha\delta_{\alpha,-\beta}f(0)g(0)c, \tag{3.1}$$

for $\alpha,\beta\in\mathbb{Z}$, $f(t),g(t)\in\mathbb{F}[t]$, where the prime stands for the derivative $\frac{d}{dt}$.

We denote

$$L_{\alpha,i}=x^\alpha t^{i+1} \text{ for } \alpha\in\mathbb{Z}, i\geq -1.$$

Then (3.1) is equivalent to (1.1).

We always use the normal order on $\mathbb{Z}$. Denote by $M(\Lambda)$ the Verma $\mathcal{B}(\mathbb{Z})$-module with highest weight vector v_Λ. Suppose $M(\Lambda)$ is reducible. Let M' denote the maximal proper submodule of $M(\Lambda)$, and set $L(\Lambda)=M(\Lambda)/M'$, the irreducible highest weight module of weight Λ. Set

$$\mathcal{A}=\{a\in\mathcal{B}(\mathbb{Z})\,|\,av_\Lambda\in M'\} \quad \text{and} \quad \mathcal{P}=\mathcal{A}+\mathcal{B}(\mathbb{Z})_0.$$

Clearly, $\mathcal{B}(\mathbb{Z})_+\subset\mathcal{A}$, and $\mathcal{P}$ is a subalgebra of $\mathcal{B}(\mathbb{Z})$.

Lemma 3.1 *$\mathcal{P}$ is a parabolic subalgebra of $\mathcal{B}(\mathbb{Z})$, namely,*

$$\mathcal{P} \supset \mathcal{B}(\mathbb{Z})_0 + \mathcal{B}(\mathbb{Z})_+ \neq \mathcal{P}. \tag{3.2}$$

Proof. The proof of (3.2) is equivalent to proving

$$\mathcal{P} \cap \mathcal{B}(\mathbb{Z})_m \neq 0 \quad \text{for some} \quad m < 0. \tag{3.3}$$

Let n be the minimal positive integer such that $U(\mathcal{B}(\mathbb{Z}))_{-n} v_\Lambda \cap M' \neq 0$. If $n = 1, 2$, one can easily verify that (3.3) holds. Assume that $n > 2$. Then there exists a vector u of weight $-n$ in M'. Write

$$u = \sum_{\substack{1 \leq l \leq n,\, \alpha_1 \leq \cdots \leq \alpha_l \\ \alpha_1 + \cdots \alpha_l = -n}} c_{\underline{\alpha},\underline{j}} x^{-\alpha_1} t^{j_1} \ldots x^{-\alpha_l} t^{j_l} v_\Lambda \in M' \quad \text{for some} \quad c_{\underline{\alpha},\underline{j}} \in \mathbb{F},$$

where $\underline{\alpha} = (\alpha_1, \cdots, \alpha_l)$, $\underline{j} = (j_1, \cdots, j_l)$, and $j_s \leq j_{s+1}$ if $\alpha_s = \alpha_{s+1}$ for $1 \leq s \leq l-1$. Moreover, we denote

$$(\underline{\alpha}, \underline{j}) = (\alpha_1, \cdots, \alpha_l, j_1, \cdots, j_l), \quad \text{and} \quad \underline{1} = (-1, \cdots, -1) \ \ (n \text{ copies of } -1\text{'s}).$$

Claim 1 *$c_{\underline{1},\underline{j}} \neq 0$ for some $\underline{j}$.*

Write (recall (2.5))

$$u \equiv \sum_{\substack{0 < \alpha_1 \leq \cdots \leq \alpha_r, \\ \alpha_1 + \cdots \alpha_l = -n}} c_{\underline{\alpha},\underline{j}} x^{-\alpha_1} t^{j_1} \ldots x^{-\alpha_r} t^{j_r} v_\Lambda \ (\mathrm{mod}\ \mathrm{V_{r-1}}),$$

where $\underline{J} = \{(\underline{\alpha}, \underline{j}) \mid c_{\underline{\alpha},\underline{j}} \neq 0\} \neq \emptyset$. Assume that there exists $(\underline{\alpha}, \underline{j}) \in \underline{J}$ such that $\underline{\alpha} \neq \underline{1}$. Let

$$(\underline{\beta}, \underline{i}) = (1, \cdots, 1, \beta_s, \cdots, \beta_r, i_1, \cdots, i_{s-1}, i_s, \cdots, i_r)$$

be the unique maximal element in $\underline{J}$ (here we use the order defined as in (2.8)), where $s \geq 1$ and $\beta_s \neq 1$. By assumption, we have $s \neq r+1$. Then for $k \gg 0$,

$$xt^k \cdot u \equiv \sum_{\alpha'_1 \leq \cdots \leq \alpha'_r} c_{\underline{\alpha'},\underline{j'}} x^{-\alpha'_1} t^{j'_1} \ldots x^{-\alpha'_r} t^{j'_r} v_\Lambda \ (\mathrm{mod}\ \mathrm{V_{r-1}}).$$

Set

$$\underline{J'} = \{(\underline{\alpha'}, \underline{j'}) = (\alpha'_1, \cdots, \alpha'_r, j'_1, \cdots, j'_r) \mid c_{\underline{\alpha'},\underline{j'}} \neq 0\},$$
$$(\underline{\beta}, \underline{j})' = (1, \cdots, 1, \beta_s - 1, \beta_{s+1}, \cdots, \beta_r, i_1, \cdots, i_{s-1}, k + i_s - 1, i_{s+1}, \cdots, i_r)$$

such that $(\underline{\beta}, \underline{j})'$ is the unique maximal element in $\underline{J}'$. The term

$$x^{-1}t^{i_1}\dots x^{-1}t^{i_{s-1}}x^{-\beta_s+1}t^{k+i_s-1}x^{-\beta_{s+1}}t^{i_{s+1}}\cdots x^{-\beta_r}t^{i_r}v_\Lambda$$

appears in $xt^k \cdot u$ since the corresponding coefficient is $-m(\beta_s k - i_s)c_{\underline{\beta},\underline{i}} \neq 0$ for some $m \in \mathbb{N}$. Thus $\underline{J}' \neq \emptyset$ and $0 \neq U(\mathcal{B}(\mathbb{Z}))_{-n+1}v_\Lambda \cap M'$, a contradiction with the assumption. Our claim follows.

Now we can write

$$u \equiv \sum_{i_1 \leq \cdots \leq i_n} c_{\underline{i}} x^{-1}t^{i_1}\dots x^{-1}t^{i_n}v_\Lambda + \sum_{l_1 \leq \cdots \leq l_{n-1}} c'_{\underline{l}} x^{-1}t^{l_1}\dots x^{-1}t^{l_{n-2}}x^{-2}t^{l_{n-1}}v_\Lambda \ (\mathrm{mod\ V_{n-2}}),$$

for some $c_{\underline{i}}, c'_{\underline{l}} \in \mathbb{F}$, where

$$\underline{I}' = \{\underline{i} = (i_1, \cdots, i_n) \mid c_{\underline{i}} \neq 0\} \neq \emptyset.$$

For any $\underline{i}$, $\underline{i}' \in \underline{I}'$, we define $\underline{i} > \underline{i}'$ as in (2.7). Let $\underline{j} = (j_1, \cdots, j_n)$ be the unique maximal element in $\underline{I}'$. For $k \gg 0$, we have a nonzero weight vector

$$xt^k \cdot u = \sum_{i'_1 \leq \cdots \leq i'_{n-1}} d_{\underline{i}'} x^{-1}t^{i'_1}\cdots x^{-1}t^{i'_{n-1}}v_\Lambda \ (\mathrm{mod}\ V_{n-2}) \quad \text{for some} \ \ d_{\underline{i}'} \in \mathbb{F},$$

since the coefficient corresponding to

$$(1, \cdots, 1, j_1, \cdots, j_{n-2}, k + j_{n-1} + j_n - 1)$$

is

$$m(k + j_{n-1})(k + j_{n-1} + j_n - 1)c_{\underline{j}} - (2k + j_{n-1} + j_n)d_{\underline{h}} \neq 0 \quad \text{for some} \ \ m \in \mathbb{N},$$

where $\underline{h} = (j_1, \cdots, j_{n-2}, j_{n-1} + j_n)$. Thus $0 \neq U(\mathcal{B}(\mathbb{Z}))_{-n+1}v_\Lambda \cap M'$, a contradiction. Our lemma follows. $\square$

By the lemma 3.1, we have $\mathcal{P}_{-1} = \mathcal{P} \cap \mathcal{B}(\mathbb{Z})_{-1} \neq 0$. Let $f(t)$ be the monic ploynomial with minimal degree such that $x^{-1}f(t) \in \mathcal{P}$. We shall call such polynomial *charactic polynomial* (cf. [S1, KL]). Set $a = x^{-1}f(t)$. Since M' is a proper submodule, we have $b \cdot av_\Lambda = 0$ for any $b \in \mathcal{B}(\mathbb{Z})_+$. From (3.1), we have

$$[xg(t), x^{-1}f(t)]v_\Lambda = \Lambda(f'(t)g(t) + f(t)g'(t) - f(0)g(0)c) = 0 \quad \text{for all} \ \ g(t) \in \mathbb{F}[t].$$

A weight $\Lambda \in \mathcal{B}(\mathbb{Z})_0^*$ is described by the *central charge* $c = \Lambda(c)$ and its *label* $\Lambda_i = \Lambda(t^i)$ for $i \geq 0$. We introduce the *generating series*

$$\Delta_\Lambda(z) = c + \sum_{i=0}^{\infty} \frac{z^{i+1}}{z!}\Lambda_i = c - \Lambda(ze^{zt}).$$

From [S1] and the above arguments, we obtain the following theorem.

Theorem 3.2 *The following conditions are equivalent:*

(1) $M(\Lambda)$ *is reducible.*

(2) $\mathcal{P}_{-1} \neq \{0\}$.

(3) $\Delta_\Lambda(z)$ *is a quasipolynomial.*

(4) $L(\Lambda)$ *is quasifinite.*

Now Theorem 1.1(1) follows from Theorem 3.2.

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